9

How Much Fit, How Much Residuals

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How Much Fit, How Much Residuals

The purpose of this chapter is to begin to deal with the question of how we can usefully assess the sizes -- absolute or relative -- of the various pieces into which we have broken down a body of data. The kind of piece that continues to concern us is one that provides a number to contribute to each observation.

9A. How much typical, how much spread?

The simplest separation of observed values into f i t p l u s r e s i d u a l s occurs when we have a single batch of values, find a single typical, and then find residuals from that typical. Most of the concepts and approaches that are needed in quite general situations arise naturally in this special simple case. Thus it will help us to look at this case in considerable detail and see, for example, just what we should ask of a "size for the residuals".

Let us first look at the subcase where the fitted typical is the batch mean. Here, as elsewhere, there are three properties that we would like to have:

- the measure of size should lead to an intuitive understanding of "about how large" the corresponding values are.

- the arithmetic needed to find the measure should be relatively easy.

- the size of the combination of two parts should be simply related to the sizes of the parts, at least approximately.

As usual, we will have to seek appropriate compromises in choosing useful measures of size.

A simple measure of size is the t y p i c a l s q u a r e of a set of values, defined as

$$\text{typical square} = \frac{\text{sum of squares of individual values}}{\text{count of values}}$$

It meets the three criteria fairly well:

- It is very easy to go from a typical square to "about how large" -- all we need to do is to find the square root of the typical square -- which might be called the (root-mean-square) typical size or the RMS typical size.

- So long as we have only a few values to deal with, this measure is not too hard to compute.

- As we shall soon see, for the separation into mean and residuals from the mean, the size of the whole is neatly (and exactly) related to the sizes of the parts.

Thus the typical square works quite well in this situation.

Consider the following set of four values -- as given and as divided into a mean plus residuals:

$$3 = 5 + (-2)$$
$$4 = 5 + (-1)$$
$$5 = 5 + (0)$$
$$8 = 5 + (3)$$

The corresponding squares, sums of squares and typical squares are:

	9	25	4
	16	25	1
	25	25	0
	64	25	9
Sum	114	100	14
Typical	28.5	25	3.5

Notice that 114 exactly equals 100 + 14 and that 28.5 exactly equals 25 + 3.5. This last is an instance of the general exact result

(Typical square of values) = (Square of mean)
+ (Typical square of residuals)

This relation is surely as simple a relation among measures of size as we could hope for, whether we think in terms of typical squares, as above, or whether we think in terms of RMS typical sizes (= $\sqrt{\text{typical squares}}$), for which the relation is

$$(\text{RMS typical size of values})^2 = (\text{mean})^2 + (\text{RMS typical size of residuals})^2$$

(Here, as noted above, "RMS" stands for root mean square.)

However we choose to describe things in detail, the situation, as we have described it, has the simple structure:

- a size for the typical, which turns out to be a simple function of the typical.
- a size for the residuals, which can be regarded as either a measure of spread or a simple function of such a measure.
- a convenient relationship, here exact, between the sizes of the parts and the size of the whole.

Now that we are clear about numbers to represent sizes, we need to turn to pictures. It is natural, as well as very alliterative, to represent typical squares by graphical squares -- squares whose area is proportional to the numerical value of the typical squares. As a bonus, of course, the sides of these squares are proportional to the typical sizes, when defined as the square roots of the typical squares.

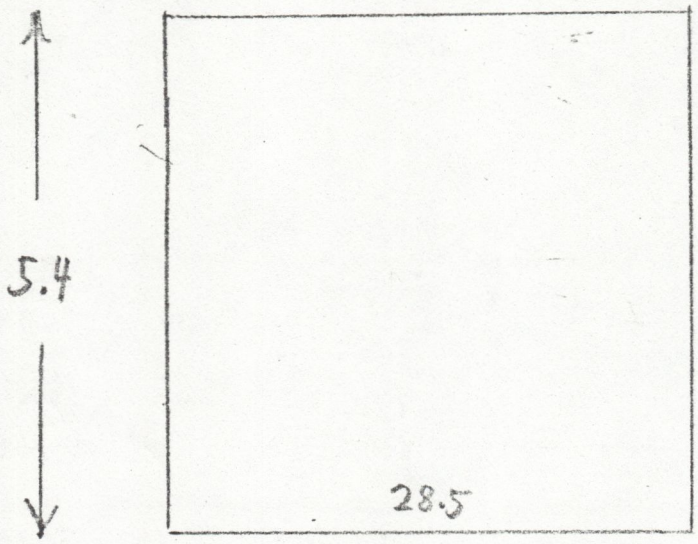
View **1** shows the result for the little example we have been carrying along. The effectiveness of this sort of pictorial presentation is quite clear.

* * *

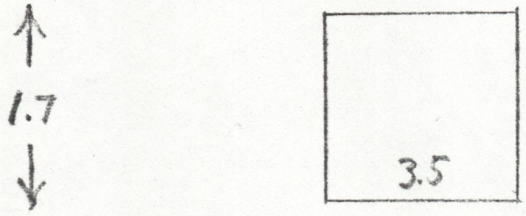
Typical squares for residuals about means have one important property -- they pay very considerable attention to far-out values. This is sometimes helpful. At other times we should like to avoid such attention. To do this it is handy to:

View 1 of chapter 9

Pictorial view of the toy example



the typical



the residuals

- turn to some measure of spread that is reasonably unaffected by far-out values, plausibly the midspread
- and to convert it into a measure of size that is compatible with a typical square.

The natural conversion is to use

$$\text{midsquare} = (\text{factor})(\text{midspread})^2$$

In chapter 3 we used a factor of 1/2 for batches of all counts. We would now like to do a little better than this. A simple choice of factor, one that works quite well, is this:

count	factor
any even number	1/2
4k-1	(2k-1)/4k
4k+1	(2k+1)/4k

Here k is any integer. Thus the factor is 1/4 for count = 3 and 3/4 for count = 5.

If we want to compare our midsquares with typical squares, we can get a somewhat more useful comparison by inserting another factor and using t y p i c a l m i d s q u a r e s where:

$$\text{typical midsquare} = \left(1 - \frac{1}{\text{count}}\right)\text{midsquare}$$

In our example we have:

Count	4
Upper hinge	6.5
Lower hinge	3.5
Midspread	3
$(\text{Midspread})^2$	9
Midsquare	$4.5 = \frac{1}{2}(9)$
Typical midsquare	3.4

The midsquare of 4.5 is thus reasonably compatible with the typical square of 3.5. The typical midsquare of 3.4 is strikingly close to the typical square (for residuals, of course) of 3.5. View 2 shows both typical squares and typical midsquare pictorially for our toy example of four values.

Had the typical midsquare been very much less than the typical square we would have suspected the presence of far-out values or, at the very least, a tendency to a straggling long-tailed appearance in the pattern of the residuals.

Conversely, had the typical midsquare been much larger than the typical square we would have expected a pattern of residuals that was blocky and broad-shouldered.

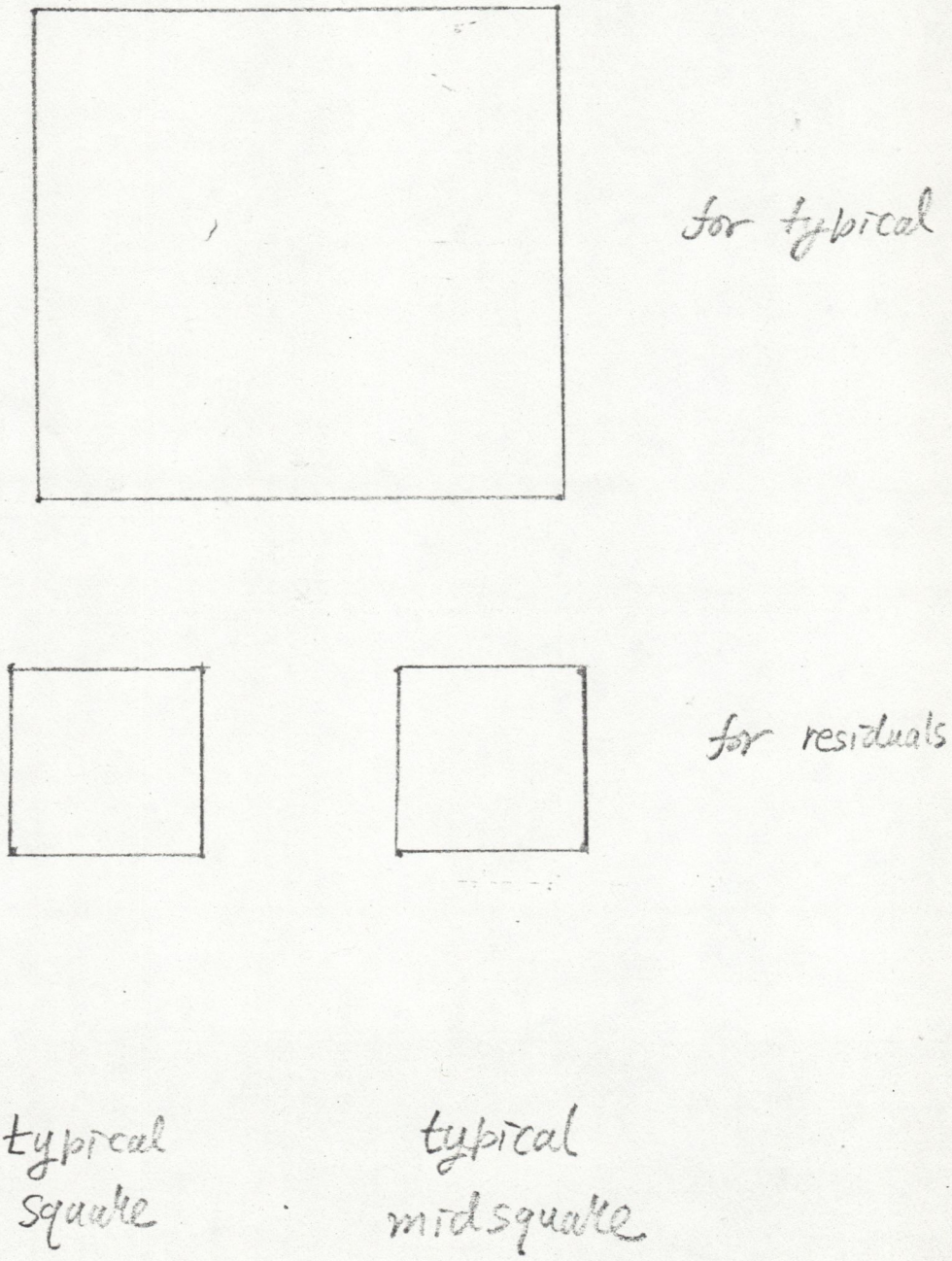
View 3 shows, schematically, five examples of such situations.

9B. Midmean residuals

Means are usually not the best typicals from which to calculate residuals. We have seen that the midmeans are more generally useful. What if we were to find typical squares

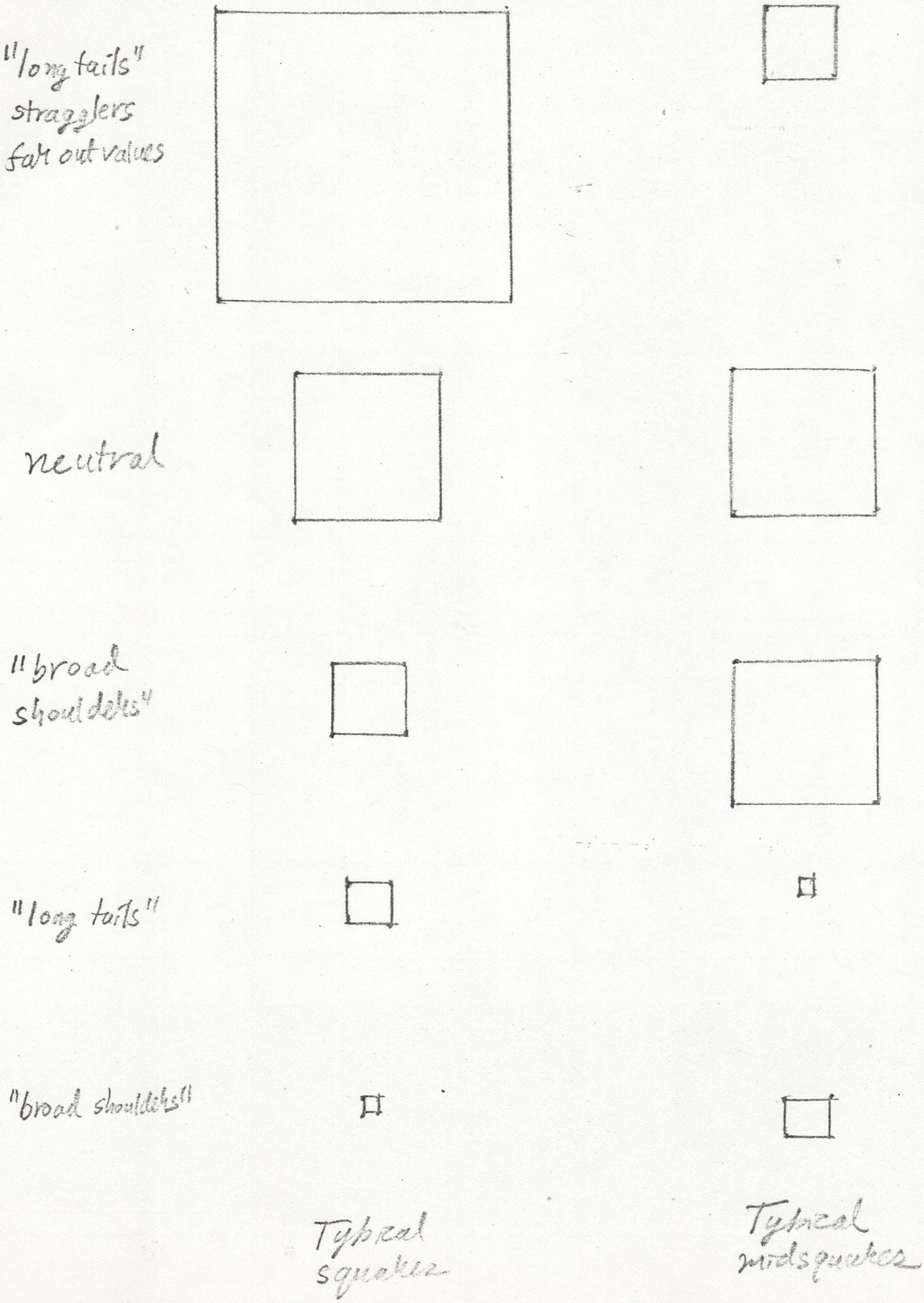
View 2 of chapter 9

The example of plot 1 expanded to show both sizes of residuals



View 3 of chapter 9

Five examples of the interpretation of the relative sizes of typical squares and mean squares



for our small example when residuals are taken from midmeans? The individual breakdowns are:

3	=	4.5	+	(-1.5)
4	=	4.5	+	(-0.5)
5	=	4.5	+	(+0.5)
8	=	4.5	+	(+3.5)

The corresponding squares, sums of squares, and typical squares are:

	9	20.25	2.25
	16	20.25	.25
	25	20.25	.25
	64	20.25	12.25
Sum	114	81	15.00
Typical	28.5	20.25	3.75

Two things are now to be noted:

- the typical squares for the parts do not add up to the typical square for the whole.

- the typical square for the residuals from the midmean is larger than the typical square for the residuals from the mean.

These same two things occur generally. We need to understand some of their implications and learn how to avoid some of the problems they raise.

Let us look at a more extreme situation, as illustrated in table 4. Changing the typical from a mean of 27 to a midmean of 13 has, in this specific example, made all but

table 4 of chapter 9

A Simple Example of Separations into Typical and Residuals

a) DATA as MEAN plus RESIDUALS

10	=	27	+	(-17)
11	=	27	+	(-16)
12	=	27	+	(-15)
12	=	27	+	(-15)
13	=	27	+	(-14)
15	=	27	+	(-12)
17	=	27	+	(-10)
126	=	27	+	(99)

b) TYPICAL SQUARES

for mean: 729 = 27^2

for residuals: 1404.5 = 11236/8

for data: 2133.5

c) DATA as MIDMEAN plus RESIDUALS

10	=	13	+	(-3)
11	=	13	+	(-2)
12	=	13	+	(-1)
12	=	13	+	(-1)
13	=	13	+	(0)
15	=	13	+	(2)
17	=	13	+	(4)
126	=	13	+	(113)

d) TYPICAL SQUARES

for midmean: 169 = 13^2

for residuals: 1600.5 = 12804/8

for data: 2133.5 (still the same)

e) COMPARISON OF INDIVIDUAL SQUARES

	Residuals from mean	Residuals from midmean
	289	9
	256	4
	225	1
	225	1
	196	0
	144	4
	100	16
	9801	12769
Sums:	11236	12804

(Cont'd)

table 4 (Cont'd)

f) COMPARISON OF MAGNITUDES OF RESIDUALS

	17	3
	16	2
	15	1
	15	1
	14	0
	12	2
	10	4
	99	113
Sums:	198	126

g) MIDSPREAD and MIDSQUARES

Midspread = 4.5 = (-11) - (-15.5) = (3) - (-1.5) = 16-11.5

$$\text{Midsquare} = 10.12 = \frac{(4.5)^2}{2}$$

$$\text{Typical midsquare} = \left(1 - \frac{1}{8}\right)(10.12) = 8.86$$

one of the residuals much smaller. The remaining residual has, however, become enough larger to make the sum of squares of residuals larger, rather than smaller. This is so because the square of 113 is so much larger than the square of 99. A similar thing does not, however, happen to the sum of magnitudes of residuals, as panel (f) of table 4 illustrates. The difference between 99 and 113, when not squared, is not enough to outweigh the other seven decreases.

In this example, the typical midsquare was 9, while the typical square around the mean (for residuals from the mean) was 1404. These summary values give a clear, extremely strong, indication of straggling, long tails, or one or more far-out values. This is just what they should do in view of the far-out residual of +99.

There is a sense in which calculating a typical square for the residuals from the midmean is calculating a size that combines some typical with all the residuals. For, as table 5 illustrates, we can match the increase -- 196 -- in typical square in passing from residuals about the mean to residuals about the midmean with the typical square for the mean of these latter residuals, also 196, which is the square of "midmean minus mean". This observation suggests that we will do well to base our measures of sizes of r e s i d u a l s on spreads -- on measures not affected by a uniform shift.

To implement this, we are going to take the typical midsquare and something like the typical square a b o u t t h e m e a n as our standard measures of squared size

table 5 of chapter 9

Breakdown of Residuals from the Midmean

a) RESIDUALS FROM THE MIDMEAN (in table 1) as DATA, and THEIR BREAKDOWN INTO MEAN AND RESIDUALS

-3	=	14	+	(-17)
-2	=	14	+	(-16)
-1	=	14	+	(-15)
-1	=	14	+	(-15)
0	=	14	+	(-14)
2	=	14	+	(-12)
4	=	14	+	(-10)
113	=	14	+	(99)

b) INDIVIDUAL SQUARES, SUMS OF SQUARES, TYPICAL SQUARES

	(about midmean)		(about mean)
	9	196	289
	4	196	256
	1	196	225
	1	196	225
	0	196	196
	4	196	144
	16	196	100
	12769	196	9801
Sum	12804	1568	11236
Typical	1600.5	196	1404.5

c) NOTE

$$1600.5 = 196 + 1404.5$$

for residuals. Between them, two such measures will usually tell us what we want to know. (Sometimes one will suffice, either because the two agree, or because we have some very specific use for our size.)

9C. q-typical squares and more general examples

The only major disadvantage of typical squares about means, so long as we do not wish to omit "far-out" values from our measure of size, is the arithmetic effort which is involved for larger batches. Fortunately -- at least so far as exploration is concerned -- we do not need to have great precision in our typical squares. A typical square of 35 is much the same as one of 37 and hardly different from one of 45. After all, typical sizes of 5.9, 6.1 and 6.5 are all pretty well described as about 6. (Here $5.9 = \sqrt{35}$, $6.1 = \sqrt{37}$, and $6.5 = \sqrt{45}$.) Accordingly, we can do quite well enough with approximate typical squares, even if they are fairly rough approximations.

When the batch of values considered is reasonably well centered around zero -- or when we have agreed to use the typical square around the mean -- a simple approximation is often useful.

For batches whose counts lie between 2 and 100, the

q u i c k t y p i c a l s q u a r e , defined by:

$$\frac{(\text{range})^2}{2 + \text{count}} \qquad \text{for counts from 2 to 10,}$$

$$\frac{(\text{next range})^2}{1 + 0.4(\text{count})} \qquad \text{for counts from 10 to 20,}$$

$$\frac{(2\text{ in})^2}{1.5 + 0.2(\text{count})} \qquad \text{for counts from 20 to 50,}$$

$$\frac{(3\text{ in})^2}{3.8 + 0.1(\text{count})} \qquad \text{for counts from 50 to 100.}$$

is often a good substitute for the typical square (about the mean). Here:

range = difference between highest and lowest
next range = difference between 2nd highest and 2nd lowest
2 in = difference between 3rd highest and 3rd lowest
3 in = difference between 4th highest and 4th lowest

For compactness we will often refer to this as the "q-typical square".

Let us begin with our simple examples. For 3, 4, 5, 8 we find a q-typical square of $5^2/6 = 4.2$, to be compared with a typical square of 3.5. For 10,11,12,12,13,15,17,118 we find a q-typical square of $(108)^2/10 = 1166$, to be compared with a typical square of 1404. Even for rather extreme batches, the agreement is fairly good.

* * *

Let us next look at some more complex examples, in which the observations have been broken up into more than two pieces. Table 15 of chapter 4 set out the essentials of an elementary analysis of data crossing three treatments with six blocks. The result is set out, in elementary analysis form, in table 6, which also sets out the q-typical and typical squares -- and also the typical midsquares.

While not all of the approximations of q-typical square to typical square are strikingly close, 2.7 instead of 2.3 being one example, there is little doubt about the relative sizes of the various pieces, namely, from larger to smaller:

- the general typical (~30)
- treatment adjustments (~2.5)
- residuals (~0.7)
- block adjustments (~0.1)

Indeed, the detailed sizes are clearly well enough defined for most purposes. View 7 shows the results pictorially.

Let us turn to another example.

Table 4 of chapter 5 gives an elementary analysis of Daniel's thread weight data in its panel (c). Table 8 repeats the elementary analysis and gives the q-typical and typical squares -- and also, again, the typical midsquares.

Again we have quite similar results, with the summary in terms of declining importance running:

- general typical (~1700)
- end group adjustments (~.9)
- bobbin adjustments (~.25)
- residuals (~.1)

table 6 of chapter 9

The Sugar Beet Data of Chapter 4
(see, for instance, tables 5 and 15 of that chapter;
stripe adjustments now adjusted to exact midmean zero)

a) ELEMENTARY ANALYSIS

	none	PO_4	PO_4+NO_3	Treat
1	-.55	1.35	-.19	-.23
2	-.75	.08	.44	-.23
3	1.57	-.25	-.60	-.42
4	.57	-.91	.09	.55
5	-.08	.88	-.56	.27
6	-.17	-1.06	1.75	.21
Blocks	-2.21	0.15	1.46	5.44

b) APPARENT SIZES

Table or Stripe	q-Typical Square	Typical Square	Typical Midsquare
Residual	.75	.68	.61
Treatment adjustments	2.69	2.34	2.25
Block adjustments	.12	.12	.10
General typical	-	29.59	-

c) EXAMPLES

$1.57 - (-.91) = 2.48 \qquad 2.48^2 = 6.15 \qquad 6.15/8.2 = .75$

$1.46 - (-2.21) = 3.67 \qquad 3.67^2 = 13.47 \qquad 13.47/5 = 2.69$

$.55 - (-.42) = .97 \qquad .97^2 = .94 \qquad .94/8 = .12$

For residuals:

Upper hinge = .57, lower hinge = -.56, midspread = 1.13

midsquare = $\frac{1}{2}(1.13)^2 = .64$, $\left(1 - \frac{1}{18}\right).64 = .61$ = typical midsquare

View 7 of chapter 9

Pictorial presentation of the analysis of table 6

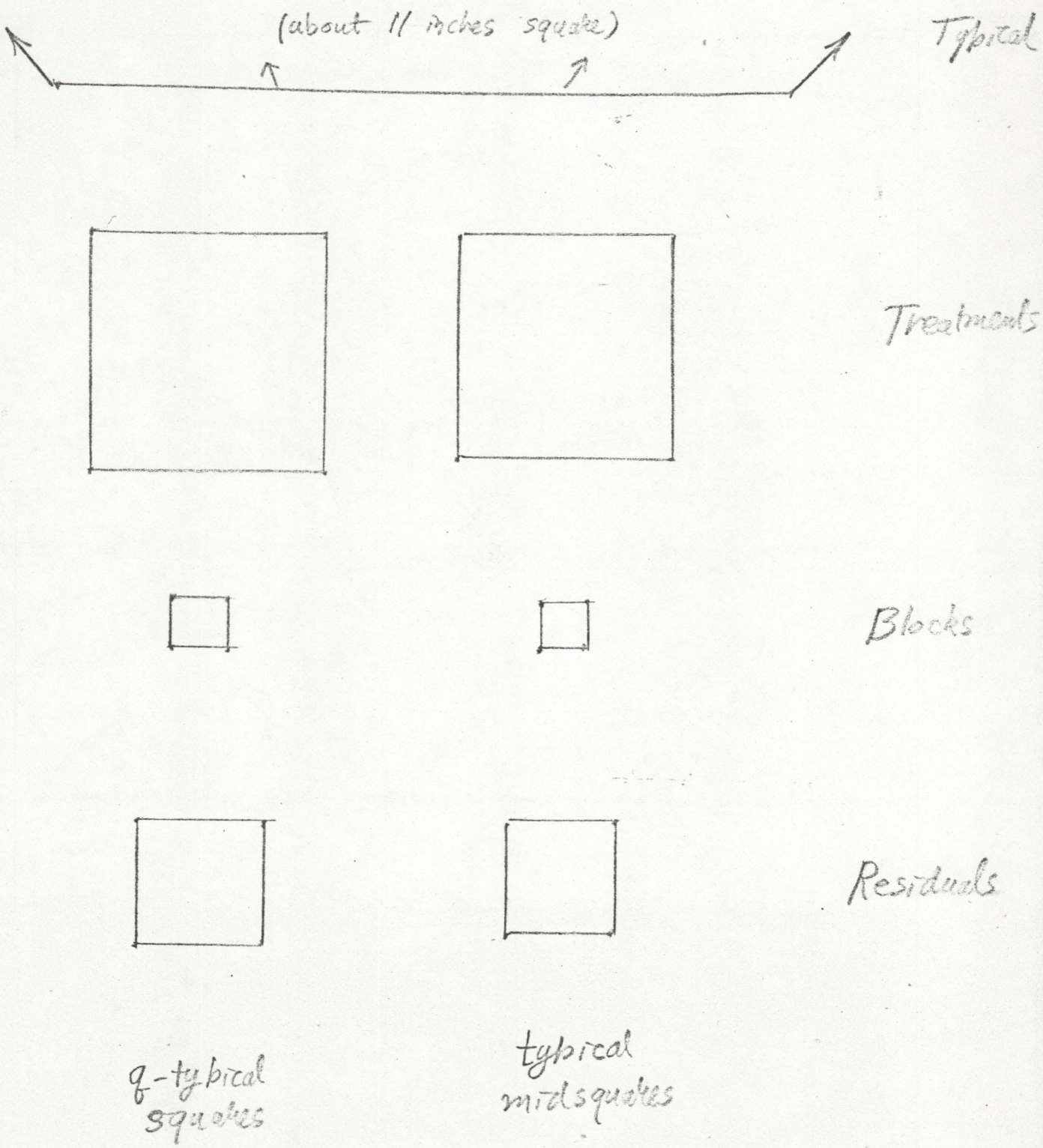


table 8 of chapter 9

The Thread Weight Data of Chapter 5
(see, for instance, tables 1,2,3,4 of that chapter)

a) ELEMENTARY ANALYSIS

	1	2	3	4	Bobbins
A	.27	-.51	-.06	-.02	-2.10
B	.03	.21	.06	-.26	-.02
C	-.32	.26	-.30	.33	.48
D	-.36	.29	-.09	.16	.32
E	.30	-.93	.23	-.19	-.29
End groups	-1.08	.10	-.10	.13	41.34

b) APPARENT SIZES

Table or Stripe	q-Typical Square	Typical Square	Typical Midsquare
Residual	.07*	.11	.13
End group adjustments	.95	.96	.22
Bobbin adjustments	.24	.30	.19
General typical	-	1708.99	-

*Using "next"; we get .08 if we use "2 in".

c) EXAMPLES

$.30 - (-.51) = .81 \qquad .81^2 = .656 \qquad .656/9 = .073$

$.29 - (-.36) = .65 \qquad .65^2 = .422 \qquad .422/5.5 = .077$

$.48 - (-2.10) = 2.58 \qquad 2.58^2 = 6.656 \qquad 6.656/7 = .95$

$.13 - (-1.08) = 1.21 \qquad 1.21^2 = 1.464 \qquad 1.464/6 = .24$

For end group adjustments:

Upper hinge = .32, lower hinge = -.29, midspread = .61
$5 = 4 \cdot 1 + 1$, $(2 \cdot 1 + 1)/4 \cdot 1 = 3/4$

Midsquare $= \frac{3}{4}(.61)^2 = .28$. $\left(1 - \frac{1}{5}\right)(.28) = .22$ = typical midsquare

The existence of a far-out value at -.93 has pushed apart the various squares for residuals somewhat, namely:

q-typical square using "next range" = .073

q-typical square using "2 in" = .077

typical square = .11

Likewise the straggling character of one end-group adjustment has separated the corresponding typical midsquare of .22 from the typical squares of .95 and .96.

Plot 9 shows the results pictorially.

9D. Striped q-typical squares

When dealing with simple batches, we have chosen to take explicit notice of the fact that residuals have had a typical value subtracted away. We did this by agreeing to use measures of spread, first typical squares about the mean and then q-typical squares.

In dealing with residuals in a two-way table, we have not yet taken explicit note of the fact that more than one typical has been removed -- that separate typicals have been removed from each row or from each column (indeed from both). We may do this if we wish by calculating quick squares for the separate stripes of one kind or the other and then taking the mean of the results. Table 10 illustrates the calculation, also giving the results of taking the mean of the range before squaring (in the table labelled as shortcuts).

We had already noted that straggling of values, especially the presence of one far-out value, produces a

Plot 9 of chapter 9

Pictorial view of the analysis of table 8

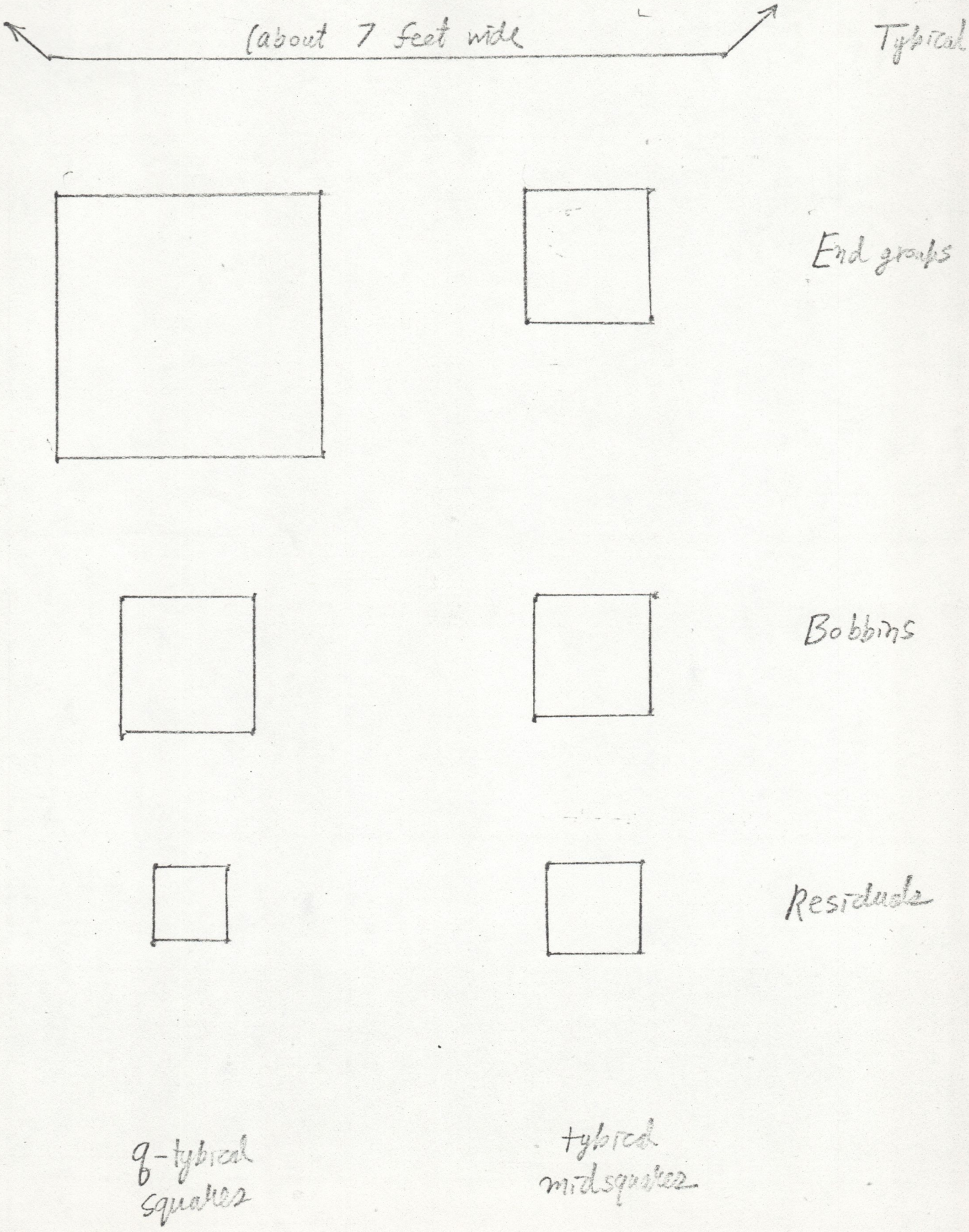


table 10 of chapter 9

Striped q-Typical Squares for the Residuals of table 8

a) THE RESIDUALS

.27	-.51	-.06	-.02
.03	.21	.06	-.26
-.32	.26	-.30	.33
-.36	.29	-.09	.16
.30	-.93	.23	-.19

b) THE ROWS REARRANGED

				Range	$(\text{Range})^2/6$
-.51	-.06	-.02	.27	.78	.101
-.26	.03	.06	.21	.47	.037
-.32	-.30	.26	.33	.65	.070
-.36	-.09	.16	.29	.65	.070
-.93	-.19	.23	.30	1.23	.252
			Mean	.76	.106

c) THE COLUMNS REARRANGED

	-.36	-.93	-.30	-.26	
	-.32	-.51	-.09	-.19	
	.03	.21	-.06	-.02	
	.27	.26	.06	.16	
	.30	.29	.23	.33	Mean
Range	.66	1.22	.53	.59	.75
$(\text{Range})^2/7$	.062	.213	.040	.050	.091

d) SHORTCUTS USING MEAN RANGE

For rows $(.76)^2/6 = .096$

For columns $(.75)^2/7 = .080$

e) UNSTRIPED VALUES (from table 8)

Typical square = .11

q-typical squares = .073 or .077

moderate but rather definite difference between the typical and q-typical squares for these residuals. We can now insert intermediate steps as follows:

typical square = .11

striped q-typical squares = .106 or .091

shortcut for same = .096 or .080

q-typical squares = .077 or .073

In this example, then, the striped q-typical squares seem to be closer to the typical square than are the q-typical squares themselves. There seems to be no reason for not using striped q-typical squares, rather they seem to be quite useful. We will regard their use, either with or without shortcuts, as one of our standard techniques.

9E. Types of summary

Let us consider various ways of stating or summarizing the result of an analysis into several pieces, say an elementary analysis by rows and columns. One extreme is to give the elementary analysis itself together with a list of apparent sizes. Panels (a) and (b) of table 8, taken together, provide an example. For reasonably small two-way tables, up to 10x10 or so, this seems to be often a reasonable solution.

At the other extreme, one that is usually far from being fully satisfactory, we can give only the list of squared sizes. Panel (b) of table 8, above, offers an example. A list of sizes alone is rarely enough to tell us what is going on.

If we are to make a useful compromise -- as we may wish to with small or moderate tables, and may have to with large tables, we can begin by giving the list of squared sizes. The question is what should we add to this. The simplest useful things to add seem to be information about a few of the highest and lowest values. An example with "few" = 2 is set out in table **11**, which summarizes the same analysis. The type of summarization set out here seems to be a reasonable choice when the full analysis is not appropriate.

It will, of course, usually be desirable to supplement this summary with a plot of residuals against plotting values, as in chapter 6.

9F. $(a+b)^2 = a^2+2ab+b^2$, and what about 2ab?

We are all familiar with the identity

$$(a+b)^2 = a^2 + 2ab + b^2$$

If we take two pieces -- of the two or more into which we have broken down some data -- and add the two together, the typical square for the sum of pieces is the arithmetic mean of the square of the sums of corresponding entries. If we apply the identity to each entry in turn, and use the simple fact that the arithmetic mean of a sum is the sum of the arithmetic means, we see that:

typical square of a+b = typical square of a
+ 2·typical product
+ typical square of b

table 11 of chapter 9

A Simple Summary of the Elementary Analysis of table 8

Table or stripe	Squared size		Lowest and highest two**				Size*
	Typical q-square*	Typical midsquare	Low	Next	Next	High	
Typical	1709			[41.3]			[41.3]
End groups	.94	.22	-2.10	-.29	.32	.48	(.47)
Bobbins	.24	.19	-1.08	-.10	.10	.13	(.44)
Residuals	.08	.13	-.93	-.51	.30	.33	(.28)

* Square root of smaller of typical q-square and midsquare.

** Identification given below.

Table or stripe	Identification of lowest and highest two			
	Low	Next	Next	High
End groups	1 to 5(A)	21 to 25(E)	16 to 20(D)	11 to 15(C)
Bobbins	1	3	2	4
Residuals	2E	2A	1E	4C

where

$$\text{typical product} = \frac{\text{sum of } a_i b_i}{\text{\# of terms}}$$

The condition that the typical square of the sum be the sum of the typical squares is thus that the t y p i c a l p r o d u c t v a n i s h e s .

Let us go back to our first example

data = mean + residuals from mean

Here the typical product is proportional to

sum of [(mean)x(residuals from mean)]

Since the mean is the same for each entry, we can move this constant factor outside the sum, finding

(mean)x(sum of residuals from mean)

Since the sum of residuals from the mean vanishes -- this can be taken as a definition of the mean -- this whole expression vanishes, as does the typical product, thus accounting for the identity we have already seen

typical square of data = $(\text{mean})^2$
+ typical square of residuals from mean

If we have a complete two-way table, and make an elementary analysis b y m e a n s , we find four parts:

- a common typical.
- row adjustments.
- column adjustments.
- residuals.

We shall leave as problems to the reader the verifications that the typical product between each pair of these parts vanishes. Since they do vanish, the typical square for any sum of 2, 3, or 4 of these parts is the sum of the typical squares of these parts.

This relation is only e x a c t when both:

- we deal with typical squares.
- the elementary analysis is by means.

Let us look at an example to see what happens if we look at q-typical squares or typical midsquares, but stick to an elementary analysis by means. Table 12 gives the elementary analysis set out in table 16 of chapter 5, and the various squared sizes of the parts.

Table 13 deals with the sum of row adjustments and column adjustments. The typical square for the sum only differs from the sum of the typical squares because of rounding errors. The q-typical squares do not match perfectly, but in view of the difference in the q-typical square itself due to the choice of "next range" or "2 in" we can give little attention to the difference. The typical midsquare comes to almost twice the sum of the typical midsquares. This is a large discrepancy. Some, but far from all, of this discrepancy

table 12 of chapter 9

The Elementary Analysis of table 16 of chapter 5 and the Associated Typical Values

a) THE DATA

	1	2	3	4	Bobbin
A	38.43	38.83	39.08	39.35	
B	40.27	41.63	41.28	41.19	
C	40.42	42.18	41.42	42.28	
D	40.22	42.05	41.47	41.95	
E	40.27	40.22	41.18	40.99	
Ends					

b) THE ELEMENTARY ANALYSIS BY MEANS

	1	2	3	4	Bobbin
A	.33	-.33	.01	.02	-1.82
B	.00	.30	.04	-.31	.35
C	-.34	.36	-.31	.29	.84
D	-.38	.39	-.10	.12	.68
E	.43	-.68	.37	-.08	-.08
Ends	-.82	.24	.15	.41	40.74

c) TYPICAL SQUARES FOR THE PARTS

Individuals	Sum	Typical	(for)
$(-.82)^2 + (.24)^2 + (.15)^2 + (.41)^2 =$	.9206	.2302	(Columns)
$(-1.82)^2 + (.35)^2 + \ldots + (-.08)^2 =$	4.6093	.9219	(Rows)
$(.33)^2 + (.01)^2 + \ldots + (-.08)^2 =$	1.9429	.0972	(Residuals)
$(40.74)^2 =$	1659.7476	1659.7476	(Typical)

d) q-TYPICAL SQUARES FOR THE PARTS

(Columns) .2512 = 1.5129/6
(Rows) 1.0108 = 7.0756/7
(Residuals) .0659 = $(.39 - (-.38))^2/9$ (from "next range")
.0891 = $(.36 - (-.34))^2/5.5$ (from "2 in")

(Cont'd)

table 12 (Cont'd)

e) MIDSQUARES FOR PARTS

	Midsquare	Typical midsquare
(Columns)	$(1/2)(.66)^2 = .2178$	.1634
(Rows)	$(3/4)(.76)^2 = .4332$	.3466
(Residuals)	$(1/2)(.625)^2 = .1953$	.1855

table 13 of chapter 9

Behavior of Sum of Row and Column Adjustments
from table 12

a) SUM OF ROW AND COLUMN ADJUSTMENTS

-2.64	-1.58	-1.67	-1.41
-.47	.59	.50	.76
.02	1.08	.99	1.25
-.14	.92	.83	1.09
-.90	.16	.07	.33

b) TYPICAL SQUARES

for sum in a: 1.1521

for row adj: .9218
for column adj: .2302
1.1520

difference = -0.0001, due to rounding to two decimals.

c) q-TYPICAL SQUARES

for sum in a using next range: $(2.76)^2/9 = .8464$

for sum in a using 2 in: $(2.66)^2/5.5 = 1.2865$

for row adj: 1.0108
for column adj: .2512
1.2620

difference = -.41 or +.03 indicates quality of approximation

d) MIDSQUARES

	Midsquare	Typical Midsquare
for sum in a	$(1.56)^2/2 = 1.2168$	1.1560
for row adj:	(.4332)	.3466
for column adj:	(.2178)	.1634
	(.6510)	.5100
difference =	.57	.65

(differences indicate quality of approximation.)

(Cont'd)

table 13 (Cont'd)

e) NOTE: The hinges for the sum are unusually loose, since they are given by:

$$-.685 = \frac{-.47 - .90}{2}$$

$$.875 = \frac{.83 + .92}{2}$$

The range of the body is 1.30, that of the inner values of the wings is 1.82. Observe that $\frac{1}{2}(1.30)^2$ is only .845, while $\frac{1}{2}(1.82)^2 = 1.65$.

is reasonably associated with the looseness of the lower hinge of the sum. Why should there be such a difference?

We shall postpone discussion of this question to the next section.

Moving on to a different sum of two parts, table 14 combines residuals and column adjustments. In this instance q-typical squares are quite closer to being additive, while typical midsquares are almost as close.

Table 15 continues the examination, summing residuals and row adjustments. Again the approximations are either within a reasonable range of variation, or quite close.

Table 16 sums all three of residuals, row adjustments, and column adjustments. Its entries are, of course, just the residuals of the original data from a single mean. The comparisons of each square for the sum with the sum of the same kind of squares for the component behave roughly as before.

The position which this example suggests to us can be summarized as follows:

- when the elementary analysis is by means, the typical square for sums is very closely the sum of the typical squares for the components -- so closely that it is feasible to compute one from others by sums or differences.

- under the same conditions, a q-typical square is likely to be not quite so closely the sum of the q-typical squares for the components -- it is likely to be close

table 14 of chapter 9

Behavior of Sum of Residuals and Column Adjustments for table 12

a) COMBINATION OF RESIDUALS AND COLUMN ADJUSTMENTS

-.49	-.09	.16	.43
-.82	.54	.19	.10
-1.16	.60	-.16	.70
-1.20	.63	.05	.53
-.39	-.44	.52	.33

b) TYPICAL SQUARES

For sum in **a**: .3268

For residuals: .0972
For column adj: .2302
.3274

difference = -.0006 due to rounding to two decimals

c) q-TYPICAL SQUARES

for **a** using next range: $(1.79)^2/9 = .3560$
for **a** using 2 in: $(1.41)^2/5.5 = .3615$

for residuals: .0608 or .0816
for column adj: .2512
.3120 or .3328

difference: +.04 or +.03, indicates quality of approximation

d) MIDSQUARES

	Midsquare	**Typical Midsquare**
for sum in **a**:	$(.94)^2/2 = .4418$	.4197
for residuals:	(.1953)	.1855
for column adjustments:	(.2178)	.1634
	(.4131)	.3489
difference =	(+.03)	+.07

differences indicate quality of approximation.

table 15 of chapter 9

Behavior of Sum of Residuals and Row Adjustments
from table 12

a) COMBINATION OF RESIDUALS AND ROW ADJUSTMENTS:

-1.49	-2.15	-1.81	-1.80
.36	.65	.39	.04
.50	1.20	.53	1.13
.30	1.07	.58	.80
.31	-.71	.29	-.16

b) TYPICAL SQUARES

for the sum in **a**: 1.0140 ±

for residuals:	.0922
for row adjustments:	.9218
	1.0140

difference: due to rounding to two decimals

c) q-TYPICAL SQUARES

for **a** using next range: $(2.94)^2/9 = .9064$

for **a** using 2 in: $(2.87)^2/5.5 = 1.4976$

for residuals:	.0608 or .0816
for row adjustments:	1.0108
	1.0716 or 1.0924

difference = -.17 or +.41

d) MIDSQUARES

	Midsquare	Typical Midsquare
for **a**	$\frac{1}{2}(1.050)^2 = .5512$	.5236
for residuals:	(.1953)	.1855
for row adj:	(.4332)	.3466
	(.6285)	.5321
difference	(-.08)	-.01

difference shows quality of approximation

table **16** of chapter 9

Behavior of Sum of Residuals and Row Adjustments and Column Adjustments

a) SUM OF THREE PARTS: residuals, row adjustments, column adjustments

-2.31	-1.91	-1.66	-1.39
.46	.89	.54	.45
-.32	1.44	.68	1.54
-.52	1.31	.73	1.21
-.51	-.42	.44	.25

b) TYPICAL SQUARES

for the sum in **a**:	1.2442 ±
for residuals:	.0922
for row adj:	.9218
for column adj:	.2302
	1.2442

difference = , due to rounding to two decimals

c) q-TYPICAL SQUARES

	next range	**2 in**
for the sum in **a**	1.2473	1.6038
for residuals:	.0618	.0816
for row adj:	1.0108	1.0108
for column adj:	.2512	.2512
	1.3238	1.3436
difference	-.07	+.26

d) MIDSQUARES

	Midsquare	**Typical Midsquare**
for sum of three in **a**:	$(1.325)^2/2 = .8778$	.8339
for residuals:	(.1953)	.1855
for column adj:	(.2178)	.1634
for row adj:	(.4332)	.3466
difference:	(.03)	.14

enough for us to be content with calculating the q-typical squares for the individual parts only, intending to use the sum of the q-typical squares of the components in case we should ever have a use to which a q-typical square of a sum could have been put.

We shall reserve the discussion of typical midsquares to the next section.

* * *

So far in this section, we have been concerned with elementary analyses using m e a n s . We have seen earlier that elementary analyses using midmean differences are safer, usually cleaner, and generally preferable. When we change from "using means" to "using midmean differences", we expect typical and q-typical squares to get bigger and typical midsquares to get smaller. The neat exact "adding up" properties of the typical squares cannot help being destroyed. In the simplest case, where row adjustments are shifted by one constant and column adjustments by another before being added together, the typical product will be the product of these constants and need not be close to zero.

Once we have gone to safer and cleaner analyses, whether using midmean differences or the streamlined approach, what we ought to say about either q-typical squares or typical squares is what we said above about q-typical squares for elementary analyses using the mean: calculate for the individual parts, and, when and if necessary, add up for sums of parts.

With the single exception of typical squares applied to an elementary analysis by means, we have this one unified picture:

- calculate typical or q-typical squares for the smallest parts considered.
- if and when useful, find (approximate) squares for sums of parts by adding up squares for the individual parts.

9G. And what of typical midsquares

Toward the middle of the last section, table 13 illustrated a situation where the typical midsquare of a sum was about twice the sum of the typical midsquares of the components. After noticing this, we postponed, to the present section, any discussion of why and how it happened.

It is easier to illustrate and explain what is happening in a 5x5 table -- a 4x5 really behaves in a similar way, but less obviously. Table 17 gives a hypothetical 5x5 example. Here the typical midsquare for the combination of row and column adjustments is 186, about 14 times the sum of the typical midsquares of the components.

The reason for this increase is not hard to find. Two-fifths of each set of five adjustments are quite extreme, ±10 or ±15. When we turn to the 25 combinations, we find that:

- four involve both a ±10 and a ±15.
- six involve only a ±10.
- six involve only a ±15.

table 17 of chapter 9

Hypothetical Example of Combination of
Row and Column Adjustments

a) THE COMBINATION

	-10	-1	0	+1	+10
-15	-25	-16	-15	-14	-5
-2	-12	-3	-2	-1	8
0	-10	-1	0	+1	10
2	-8	1	2	3	12
15	5	14	15	16	25

b) THE COMBINED VALUES DISPLAYED

+2	5	5
+1	02456	02456
+0	8011235	011235 8
-0	532118	853211
-1	65420	02456
-2	5	5

count = 25 hinges are -8 and +8

midsquare = $\frac{3}{4}(16)^2 = 196$

typical midsquare = 186

(typical square = 132)

c) MIDSQUARES OF COMPONENTS

	Midsquare	Typical midsquare	(Typical square)
Row adj:	$\frac{3}{4}(4)^2 = 12$	10	(91.6)
Column adj:	$\frac{3}{4}(2)^2 = 3$	3	(40.4)
		13	(132)

Thus 14 out of the 25 combinations involve at least one of the extreme values not involved in the values of the midsquares of the components. Since 14/25 = 56% > 50%, we cannot expect the midsquare of the combination to evade all the values influenced by the extreme values of the components.

The possibility of such a phenomenon is usually warned to us by a notable difference between typical or q-typical square and typical midsquare for one or both components. For table 13, the row values are 1.01 and 0.35, the column values are .25 and .16. Thus some warning was given. For table 17, where the effect was greater, a much more extreme warning is given, as can be seen from the q-typical squares being 9 or 10 times as large as the typical midsquares. The possibility of such phenomena tells us something about where we should use typical midsquares.

Typical midsquares have two main purposes:

- as something to compare with a typical square in order to detect tendencies toward long tails or broad shoulders.

- as a numerical indication -- only in those instances where the typical midsquare is appreciably smaller than the typical square -- of what the typical square might become if we could remove the tendencies to long tails or straggling which appear to be present.

What has happened in tables 13 and 17 shows that a combination can show less of a tendency to long tails than do the components. Indeed, if the components have distinctively

long tails, those of the sum are likely to be shorter. If we are to be wise in looking for long tails, then, we should look where they are more likely to be revealed -- in the components. For the first purpose of typical midsquares, then, there seems to be little purpose in calculating them for sums.

A very similar argument applies to the second purpose. If we are to gain by getting rid of stragglers and long tails, we will gain most by doing this where the stragglers are most erratic and the tails are longest. This, again, will be in the components, not in the sums. Again, we want typical midsquares for the smallest parts that we have, not for sums of two or more such.

Whatever our main purpose, then, our need for typical midsquares is confined to the smallest parts.

* * *

It is interesting to notice that the phenomenon of sums behaving more blandly than the terms from which the sums are made up is not confined to long tails. The same thing can happen for broad shoulders, as table 18 shows. Again, it will usually be true that if the components tend to have notably broad shoulders, their sum will be less noticeably broad shouldered.

Sometimes this phenomenon -- sums being less unusual than their components -- is regarded as a useful property of sums. After all, it is correctly argued, this makes the behavior

table 18 of chapter 9

Behavior of Sum of Hypothetical Row and Column Adjustments
(Broad Shoulders)

a) THE COMBINATION

	-10	-9	0	9	10
-15	-25	-24	-15	-6	-5
-14	-24	-23	-14	-5	-4
0	-10	-9	0	9	10
14	4	5	14	23	24
15	5	6	15	24	25

b) MIDSQUARES

	Midsquare	Typical Midsquare	(Typical Square)
for the combination in a:	$\frac{1}{2}(20)^2 = 200$	190	(240.8)
for row adj:	$\frac{1}{2}(28)^2 = 392$	314	(168.4)
for column adj:	$\frac{1}{2}(18)^2 = 162$	130	(72.4)
		444	(240.8)

of sums more bland and predictable. There are points of view from which this remark is both correct and reasonable. For us, with our concern for exploration, the remark, though correct, is quite misleading.

So far as exploration goes, the situation is quite the reverse. We have a much better chance of identifying something new and interesting when the values we examine are NOT bland. Bland is too likely to mean unattackable. If there are long tails, perhaps we can isolate one or a few values as stragglers -- once isolated, we have a chance to find possible interpretations and look elsewhere for relevant information. If there are broad shoulders, there is a chance that we can find a rational way to divide the cases into two classes, of such a nature that one shoulder tends to come from class A, the other from class B. If we can do this, we again have a useful appearance to study.

For the further explorer, the more that the typical midsquare deviates from the typical or q-typical square the better. That components are likely to be l e s s bland is for him a blessing.

9H. Fitting tilts and bends

Besides single batches, and various more or less complete forms of the two-way table, we have considered one other class of situations where we break up the data into fit plus residual. In chapter 7, we dealt with the fitting of tilts and bends. Here, too, we can properly concern ourselves with relative sizes.

When we come to look carefully at the situation, we find it a little different from the cases we have so far considered. In the two-way situation, it can easily be quite reasonable to make row adjustments but not column adjustments for this body of data, and to reverse our field and make column adjustments but not row adjustments for another body.

Fitting a straight line can quite properly be regarded as fitting a constant part and a tilt. It is often reasonable to fit the constant part without the tilt, yet only in special situations is it reasonable to fit the tilt without the constant part.

Extending this view to include bend, the three breakdowns that we are likely to consider are:

data = (constant in x) + (first residuals)

data = (straight line in x) + (second residuals)

data = (quadratic in x) + (third residuals)

The other four possibilities -- of fitting some, but not all, of shift, tilt, and bend -- we leave to the reader. They are only likely to be useful in special circumstances.

For each of these three breakdowns there are two parts we can size, a total of six. Which of these interest us? Why?

Perhaps most often, our attention should go to the quality of fit. How much better do we fit the data as we bring more flexibility into the curve with which we are doing the

fitting? An adequate answer to this question would require deeper considerations than we are prepared to give here. (At the very least, we should try to think how closely the presently fitted curve may be expected to fit n e w data.)

If we cannot answer this question, we can answer another, one helpful in itself. How well do these various stages of fitting perform for the data to which they have been fitted? How large are the changes?

To this end, we might like to look at squared sizes for:

- the data.
- the first residuals.
- the second residuals.
- the third residuals.

We expect these values to decrease as we fit a more and more flexible curve. The question is: by how much will they decrease?

How we try to answer this question will -- and should -- depend on our purposes. Which is our purpose:

- to account for -- and remove by subtraction -- enough of the variability in y that appears associated with x to make the residuals useful for some specific purpose?
- to obtain nice clean residuals, which are free of apparent systematic effects of x?
- to fit as detailed a dependence of y on x as we can find any strong suggestion for?

While we do not have the background to treat all of these purposes fully, we can say something useful about each of them. In particular, we can say that moving down this list of purposes will ordinarily cause us to fit more flexible curves, involving more constants, rather than the opposite.

If we are merely trying to get the size of the residuals down, there are two likely possibilities:

- we have a typical square in mind, such that we will be reasonably happy if we get the typical square residual at least that small.

- having no target, we just want to get the size of the residuals down, and tend to judge the gain from fitting one more constant by the r a t i o of squared sizes before and after the additional constant has been fitted.

For these purposes, following the typical or q-typical square often seems most natural. In some situations, where a few erratic values are of little concern, it may be more natural to follow the typical midsquare instead.

If our purpose is to reach clean residuals, we need better guidance than our present background will provide. We may be able to get along with the rule of thumb which says stop somewhere between the places you would stop for the previous and following purposes.

If our purpose is to go as far as strong suggestion will take us, it is natural to look at the reduction in squared size, expressed as a difference, that comes from fitting one

more constant -- transferring one more degree of freedom from residuals to fit -- making the bundle of possible curve one step more flexible. In general, how do we expect these reductions to behave? Surely we can expect:

- a tendency to irregularity and fluctuation.
- a tendency to start with large reductions and move toward smaller ones.
- an eventual tendency to flatten out.

Plotting the log of the reduction against the serial number of the constant fitted seems often to give useful guidance as how far to go. If we can see a slant-down elbowing into a horizontal portion, we can reasonably go almost to the elbow.

Which squared size do we follow? So long as either typical midsquare or (q-)typical square continues to shrink noticeably, there is a reasonable suggestion that the constant in question could reasonably be fitted. Accordingly, it can be helpful to plot both graphs, look for both natural stopping points and go on to the more inclusive one.

* * *

Surely we should look at some examples. We can provide some from our own tilt and bend fitting. However, three or four points are unlikely to show a clear elbow, so we will do well to look at some examples where others have done the fitting, examples which go further up the scale of flexibility.

Fisher and Yates, in Example 10 of their Statistical Tables for Biological, Agricultural and Medical Research (4th ed. 1953) treat the fitting of polynomials up to the 5th degree to the difference in yield (bushels per acre) of plots 9 and 7 of Broadbalk Wheatfield for the years 1855 to 1884 (yields rounded to the nearest 0.1). The corresponding typical squares are as follows:

after fitting	typical square	reduction
constant	33.90	
		5.27
1st degree (st. line)	28.63	
		8.92
2nd degree	19.71	
		.12
3rd degree	19.59	
		.20
4th degree	19.39	
		.08
5th degree	19.31	

No plot is needed to indicate that we must stop after the 2nd degree fit -- or only after going beyond the 5th.

By fitting tilt and bend, the RMS size of the residual decreases as follows:

after fitting	RMS residual
constant	5.8
constant and tilt	5.4
constant, tilt, and bend	4.4

Thus the decrease in typical size of residual, though useful, is hardly striking.

9I. How far have we come?

Describing the general size of the terms into which we have broken up our data can be useful. Often giving "high, low, and range" suffices. But there are times when we wish to allow for the fact that, among terms "of the same size," those with more distinct values tend to have larger ranges. In looking for such measures, it is usually good to retain one property of the range -- its immunity from uniform shifts.

Typical squares (about the mean) do quite well:

- if we want to notice far-out and nearly far-out values, and

- if we do not mind the arithmetic.

For hand computation, q-typical squares -- striped when convenient -- seem to be a reasonable substitute for typical squares, one that involves much less computation.

When we do not wish to pay attention to extreme values, the midsquare (as redefined about) proves quite useful. For comparison with typical or q-typical squares (about the mean), we do better to modify our numbers slightly, forming typical midsquares.

Graphical displays naturally use squares with area proportional to typical square or midsquare.